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Response Factorization of Simply Connected Ising Lattices with Application to Bethe Lattice Spin Glasses

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The thermodynamics of a classical lattice gas in Ising form, with arbitrary interaction, is set up in entropy format, with multipoint magnetizations as control parameters. It is specialized to the case of one- and two-point interactions on a simply connected lattice; both entropy and profile equations are written down explicitly. Linear response functions are expressed in Wertheim–Baxter factorization and used to derive the Jacobian of the transformation from couplings to magnetizations. An arbitrary spin-glass coupling distribution is transformed to the corresponding magnetization distribution, whose effect on thermodynamic properties is assessed. A Gaussian coupling-fluctuation expansion diverges at sufficiently large fluctuation amplitude, suggesting the possibility of a phase transition.

KEY WORDS: Nonuniform Ising model; spin glass; Bethe lattice.

1. INTRODUCTION

Under a variety of circumstances, the magnetic interaction between atomic sites on a crystal lattice can be represented by that of a classical nearest-neighbor Ising model. However, the interaction strength J_{xy} depends as well upon the internal states of the sites involved. If the internal states change very slowly on the time scale of thermalization of the Ising spins, they may be regarded as fixed, a quenched configuration. But over a very long time period, one will see a suitably weighted ensemble of such configurations, over which thermodynamic energies and expectations must then be averaged. Alternatively, without going to a very long-time average,

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but instead to the thermodynamic limit of a very large volume, suppose one can decompose the lattice into large subdomains with identical environment but which effectively sample the internal state subdomain configurations. The result once more is that averages over these configurations are to be taken to obtain observable energies and expectations. One can create situations in which the external fields h_x are similarly distributed. Thus, quantities of interest will take the form

$$\bar{A} = \int \cdots \int A\{h_x, J_{xy}\}\, w\{h_x, J_{xy}\} \prod dh_x \prod dJ_{xy} \tag{1.1}$$

where $w\{h_x, J_{xy}\}$ is the weight (probability density) of the configuration $\{h_x, J_{xy}\}$. This describes a quenched spin glass (see, e.g., ref. 1 for a comprehensive review).

To be somewhat more explicit, denote by Ω the set of sites on the lattice and $\mathscr{S}$ the set of clusters of sites that interact, i.e., single sites and nearest-neighbor pairs in the systems that we will soon focus on. We restrict our attention to thermal equilibrium at temperature T. The lattice energy will then be taken, in units of kT, as

$$H = -\sum_{A \in \mathscr{S}} J_A \sigma_A \qquad \text{where} \quad \sigma_A \equiv \prod_{x \in A} \sigma_x \tag{1.2}$$

and of course each $\sigma_x = \pm 1$. At fixed $\{J_A\}$, the thermodynamics is determined by the partition function

$$Z = \sum_{\{\sigma_x\}} \exp\left(\sum J_A \sigma_A\right) \tag{1.3}$$

and the corresponding free energy

$$F = -\ln Z \tag{1.4}$$

(or grand partition function from a lattice gas viewpoint, with equation of state $P\Omega = -F$ for a homogeneous gas).

Equally importantly, F generates the basic multisite magnetizations

$$m_A = \langle \sigma_A \rangle = -\partial F / \partial J_A \tag{1.5}$$

as well as their correlations

$$\begin{aligned} \langle (\sigma_A - \langle \sigma_A \rangle)(\sigma_{A'} - \langle \sigma_{A'} \rangle) \rangle &= -\partial^2 F / \partial J_A\, \partial J_{A'} \\ &= \partial m_{A'} / \partial J_A \end{aligned} \tag{1.6}$$

If $\{J_A = J_A^0 + \Delta J_A\}$, where $\{\Delta J_A\}$ has a nontrivial distribution, and $\bar{F}$ is defined as in (1.1), it follows as well that

$$\bar{m}_A = -\partial\bar{F}/\partial J_A^0 \tag{1.7}$$

so that $\bar{F}$ generates the spin-glass expectations and thermodynamics. The computation of $\overline{\ln Z\{J_A\}}$ is, however, a nontrivial task (as opposed to that of an annealed spin glass, in which the time scales of internal state and spin thermodynamics reverse their order, so that only $\overline{Z\{J_A\}}$ has to be found). Many approximation techniques have been proposed, some—the replica trick in its broken symmetry form[2]—hopefully but not provably exact, and a very small number of exact results for special lattices and parameter values.[3]

In this paper, we focus on the class of lattices for which some exact solutions exact, that of simply connected lattices or open Cayley trees, or Bethe lattices if each site has the same connectivity q_x (the number of sites which are nearest neighbors). These are characterized by the existence of exactly one path connecting any two lattice vertices, and the consequent fact that excision of any site breaks up the lattice into two or more disconnected sublattices is what makes them solvable. A lattice that locally resembles a Cayley tree, i.e., the region within a few sites f of a given site is simply connected ($f = 1$ for a square lattice, $f = 2$ for a honeycomb), can be tolerably well approximated by such a lattice. Although $Z\{J_A\}$ cannot be expressed in anything like closed form for even these lattices, an inverse representation in terms of the m_A is available,[4] with entropy as the thermodynamic potential or generating function. Here, this formulation is reviewed, a new correlation factorization developed, and a spin-glass average written down in schematic form. Explicit evaluation for typical configuration weights is not obviously feasible, but an appropriate recursion relation is derived. Finally, an expansion is made about the region of fixed coupling constant, demonstrating the possibility of fluctuation-induced phase transitions in such systems.

2. THE ENTROPY FUNCTIONAL

Let us for the moment retain the general form (1.2) of the lattice energy at fixed $\{J_A\}$. The corresponding entropy, in units of Boltzmann's constant, is normally defined as

$$S = -\sum_{\{\sigma_A\}} \rho\{\sigma_A\} \ln \rho\{\sigma_A\} \tag{2.1}$$

in terms of the full lattice probability density

$$\rho\{\sigma_A\} = W\{\sigma_A\}/Z, \qquad \sum_{\{\sigma_A\}} \rho\{\sigma_A\} = 1 \tag{2.2}$$

constructed from the Boltzmann factor

$$W\{\sigma_A\} = \exp \sum_{A \in \mathscr{S}} J_A \sigma_A \tag{2.3}$$

If the control parameters $\{J_A\}$ are varied by $\{\delta J_A\}$, then

$$\delta Z = Z \sum_{A \in \mathscr{S}} \{\sigma_A\rangle \, \delta J_A \tag{2.4}$$

from which

$$\delta\rho\{\sigma_A\} = \rho\{\sigma_A\} \sum_A (\sigma_A - \langle\sigma_A\rangle) \, \delta J_A \tag{2.5}$$

and consequently

$$\begin{aligned} \delta S &= -\delta\langle \ln \rho\{\sigma_A\}\rangle \\ &= -\left\langle \sum J_{A'} \, \delta J_A \, (\sigma_{A'} - \langle\sigma_{A'}\rangle)(\sigma_A - \langle\sigma_A\rangle)\right\rangle \end{aligned} \tag{2.6}$$

Inserting (1.6), we conclude that

$$\delta S = -\sum_{A'} J_{A'} \, \delta m_{A'} \tag{2.7}$$

According to (1.6), $\partial m_{A'}/\partial J_A$ is positive definite as a matrix and hence invertible. Thus, the $\{m_A\}$ are independent and uniquely determine the $\{J_A\}$. We henceforth regard the $\{m_A\}$ as the control parameters, (2.7) then telling us that

$$J_A = -\frac{\partial S}{\partial m_A} \tag{2.8}$$

Of course, (2.8) could equally well have been oobtained by a Legendre transformation from F of (1.4) and $\{m_A\}$ of (1.5), switching from the variables $\{J_A\}$ to the set $\{m_A\}$. This implies the relation

$$S + F = -\sum_A J_A m_A \tag{2.9}$$

which one can just as easily obtain directly.

It is the quantity $S\{m_A\}$ that we seek, a functional (to use the term more usually assocaited with continuum indices) of the magnetizations $\{m_A,\ A \in \mathscr{S}\}$. Let us now specialize to the cases of present interest, in which the energetic components arise only from single sites x, or pairs of nearest neighbor sites, denoted by $\langle x, y\rangle$. We also transfer to a more standard notation by the replacement

$$\{m_x, m_{xy}, J_x, J_{xy}\} \to \{m_x, g_{xy}, h_x, J_{xy}\} \tag{2.10}$$

Furthermore, it will be convenient to introduce the joint probabilities $n_A[\alpha]$ of the sites in $A = \{x\}$ having the spin value $\alpha = \{\alpha_x\}$. Since

$$\tfrac{1}{2}(1 + \alpha\sigma) = \delta_{\sigma,\alpha} \tag{2.11}$$

this implies the general relation

$$n_A[\alpha] = \left\langle \prod_{x \in A} \frac{1}{2}(1 + \alpha_x \sigma_x)\right\rangle \tag{2.12}$$

Expanding the product as $(1/2^{|A|}) \sum_{A' \subset A} \alpha_{A'} \langle \sigma_{A'}\rangle$, one also has

$$n_A[\alpha] = \frac{1}{2^{|A|}} \sum_{A' \subset A} \alpha_{A'} m_{A'} \tag{2.13}$$

of which the subcases

$$\begin{aligned} n_x(\alpha) &= \tfrac{1}{2}(1 + \alpha m_x) \\ n_{xy}(\alpha, \alpha') &= \tfrac{1}{4}(1 + \alpha m_x + \alpha' m_y + \alpha\alpha' g_{xy}) \end{aligned} \tag{2.14}$$

will be particular useful.

Let us specialize further to simply connected lattices. Since there is just one path connecting any two sites, the notation

$$y \in (z, z') \tag{2.15}$$

meaning that y is on the path between z and z', makes perfect sense. Now any y splits the remaining lattice sites (not uniquely) into two disjoint sets A_y and B_y: $\{x\} = A_y + (y) + B_y$, on opposite sides of y: if $z \in A_y$, $z' \in B_y$, then $y \in (z, z')$. It follows that, in obvious notation

$$W\{\sigma_x\} = e^{h_y \sigma_y} W_{\sigma_y}\{\sigma_z \mid z \in A_y\}\ W_{\sigma_y}\{\sigma_{z'} \mid z' \in B_y\} \tag{2.16}$$

Consequently, e.g., using (2.12), and supposing that $z_i \in B_y$ for $i = 1,\ldots, s$,

$$\begin{aligned} &\text{if} \quad y \in (z, z_i) \quad \text{for} \quad i = 1,\ldots, s, \quad \text{then} \\ &n_{zz_1 \cdots z_s}(\alpha_z, \alpha_{z_1},\ldots, \alpha_{z_s} \mid \alpha_y) = n_z(\alpha_z \mid \alpha_y)\, n_{z_1 \cdots z_s}(\alpha_{z_1},\ldots,\alpha_{z_s} \mid \alpha_y) \end{aligned} \tag{2.17}$$

[equivalent to the extended Markov condition $n_z(\alpha_z \mid \alpha_y, \alpha_{z_1},..., \alpha_{z_s}) = n_z(\alpha_z \mid \alpha_y)$]. Suppose that Λ is a convex set of sites, i.e., contains every site on the path between any two sites, that $z \in \partial\Lambda$, the boundary of Λ, and that $y \in \Lambda$ is the unique nearest neighbor of z. Then according to (2.17),

$$n_\Lambda[\alpha] = n_z(\alpha_z \mid \alpha_y)\, n_{\Lambda - z}[\alpha - \alpha_z] \tag{2.18}$$

In the form[4]

$$n_\Lambda[\alpha] = \frac{n_{zy}(\alpha_z, \alpha_y)}{n_y(\alpha_y)}\, n_{\Lambda - z}[\alpha - \alpha_z] \tag{2.19}$$

we can nibble away at the boundary of any convex set, reducing it to a pair of sites. In particular, for the full lattice $\Omega = \{x\}$, we arrive in this fashion at

$$\rho\{\alpha_x\} = \prod_{\langle x, y\rangle} n_{xy}(\alpha_x, \alpha_y) \Big/ \prod_x n_x(\alpha_x)^{q_x - 1} \tag{2.20}$$

Inserting (2.20) into (2.1) now yields at once the desired expression

$$S = \sum_{x, \alpha} (q_x - 1)\, n_x(\alpha) \ln n_x(\alpha) - \sum_{\langle x, y\rangle \alpha, \alpha'} n_{xy}(\alpha, \alpha') \ln n_{xy}(\alpha, \alpha') \tag{2.21}$$

The profile equations then follow as well:

$$\begin{aligned} h_x &= -\frac{\partial\partial S}{\partial m_x} = \frac{1}{4} \sum_{\langle y, x\rangle, \alpha, \alpha'} \alpha \ln n_{xy}(\alpha, \alpha') - \frac{1}{2}(q_x - 1) \sum_\alpha \alpha \ln n_x(\alpha) \\ J_{xy} &= -\frac{\partial S}{\partial g_{xy}} = \frac{1}{4} \sum_{\alpha, \alpha'} \alpha\alpha' \ln n_{xy}(\alpha, \alpha') \end{aligned} \tag{2.22}$$

Since (2.22) implies that

$$\begin{aligned} &\sum_x h_x m_x + \sum_{\langle x, y\rangle} J_{xy}\, g_{xy} \\ &\quad = \tfrac{1}{4} \sum_{\langle x, y\rangle, \alpha, \alpha'} (\alpha m_x + \alpha' m_y + \alpha\alpha' g_{xy}) \ln n_{xy}(\alpha, \alpha') \\ &\qquad - \tfrac{1}{2} \sum_{x, \alpha} (q_x - 1)\, \alpha m_x \ln_x(\alpha) \end{aligned} \tag{2.23}$$

we also obtain from (2.9) the simple expression

$$F = -\tfrac{1}{2} \sum_{x, \alpha} (q_x - 1) \ln n_x(\alpha) + \tfrac{1}{4} \sum_{\langle xy\rangle, \alpha, \alpha'} \ln n_{xy}(\alpha, \alpha') \tag{2.24}$$

or alternatively

$$F = \frac{1}{2}\sum_{x,\alpha} \ln n_x(\alpha) + \frac{1}{4}\sum_{\langle x, y\rangle, \alpha, \alpha'} \ln \frac{n_{xy}(\alpha, \alpha')}{n_x(\alpha)\, n_y(\alpha')} \tag{2.25}$$

3. LINEAR RESPONSES AND CONSEQUENCES

According to (1.6), the linear response functions $\partial m_{A'}/\partial J_A$ are related to the magnetization–magnetization correlations. The inverse linear respense $\partial J_A/\partial m_{A'} = -\partial^2 S/\partial m_A\, \partial m_{A'}$ which characterizes entropy fluctuations, has, however, a much simpler mathematical structure for the nearest-neighbor simply connected lattices we are considering. Switching to the appropriate notation, and using (2.14), we now have

$$\begin{aligned}
\frac{\partial h_x}{\partial m_x} &= -\frac{q_x - 1}{4}\sum_{\alpha}\frac{1}{n_x(\alpha)} + \frac{1}{16}\sum_{\langle y, x\rangle, \alpha, \alpha'}\frac{1}{n_{xy}(\alpha, \alpha')} \\
\frac{\partial h_x}{\partial m_y} &= \frac{1}{16}\sum_{\langle y, x\rangle, \alpha, \alpha'}\frac{\alpha\alpha'}{n_{xy}(\alpha, \alpha')} \\
\frac{\partial h_x}{\partial g_{xy}} &= \frac{1}{16}\sum_{\alpha, \alpha'}\frac{\alpha'}{n_{xy}(\alpha, \alpha')}
\end{aligned} \tag{3.1}$$

for y a nearest neighbor of x, and similarly

$$\begin{aligned}
\frac{\partial J_{xy}}{\partial m_x} &= \frac{1}{16}\sum_{\alpha, \alpha'}\frac{\alpha'}{m_{xy}(\alpha, \alpha')} \\
\frac{\partial J_{xy}}{\partial m_y} &= \frac{1}{16}\sum_{\alpha, \alpha'}\frac{\alpha}{n_{xy}(\alpha, \alpha')} \\
\frac{\partial J_{xy}}{\partial g_{xy}} &= \frac{1}{16}\sum_{\alpha, \alpha'}\frac{1}{n_{xy}(\alpha, \alpha')}
\end{aligned} \tag{3.2}$$

as the only nonvanishing derivatives. These constitute a generalized set of (complete) direct correlation functions in lattice gas parlance. In addition, they have a classical Wertheim–Baxter factorization. Verifying this statement requires a bit of work.

It is convenient to establish an ordering of sites on our network, assumed finite. For this purpose, select any boundary point, say x_0, as origin (or root) and given it ordinal number 0. Therefore the sites are ordered according to their generation number, i.e., the distance to x_0, and within each generation ordered in any fashion at all—they will never have to be compared. This means that x has just one nearest neighbor $x' < x$, and $q_x - 1$ nearest neighbors $y > x$ (which satisfy $y' = x$). Now the factorization desired will take the form

$$\begin{pmatrix} \partial J/\partial g & \partial J/\partial m \\ \partial h/\partial g & \partial h/\partial m \end{pmatrix} = \begin{pmatrix} \partial J/\partial g & 0 \\ \partial h/\partial g & Q \end{pmatrix} \begin{pmatrix} (\partial J/\partial g)^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \partial J/\partial g & \partial J/\partial m \\ 0 & Q^T \end{pmatrix} \tag{3.3a}$$

where $Q_{xy} = 0$ unless

$$y = x \quad \text{or} \quad y > x \text{ is a nearest neighbor of } x \tag{3.3b}$$

The equality of (3.3a) demands that

$$QQ^T = \partial h/\partial m - \partial h/\partial g\, (\partial J/\partial g)^{-1}\, \partial J/\partial m \tag{3.4}$$

and we want to show that this can be satisfied under condition (3.3b). Inserting (3.1) and (3.2), we find that the (x, x) element if (3.4) now reads

$$\begin{aligned} Q_{xx}^2 + \sum_{\substack{(y,x)\\ y>x}} Q_{xy}^2 = {} & -\frac{1}{4}(q_x - 1) \sum_\alpha \frac{1}{n_x(\alpha)} + \frac{1}{16} \sum_{\langle y,sx\rangle} \left\{ \left[\left(\sum_{\alpha,\alpha'} \frac{1}{n_{xy}(\alpha,\alpha')} \right)^2 \right. \right. \\ & \left. \left. - \left(\sum_{\alpha,\alpha'} \frac{\alpha'}{n_{xy}(\alpha,\alpha')} \right)^2 \right] \Big/ \sum_{\alpha\alpha'} \frac{1}{n_{xy}(\alpha,\alpha')} \right\} \end{aligned} \tag{3.5}$$

while the (x, y) element for $y > x$, a nearest neighbor (it is not necessary to evaluate for $y < x$, since $QQ^T$is symmetric), becomes

$$\begin{aligned} Q_{xy} Q_{yy} = {} & \frac{1}{16} \left[\sum_{\alpha,\alpha'} \frac{\alpha\alpha'}{n_{xy}(\alpha,\alpha')} \sum_{\alpha,\alpha'} \frac{1}{n_{xy}(\alpha,\alpha')} \right. \\ & \left. - \sum_{\alpha,\alpha'} \frac{\alpha}{n_{xy}(\alpha,\alpha')} \sum_{\alpha\alpha'} \frac{\alpha'}{n_{xy}(\alpha,\alpha')} \right] \left(\sum_{\alpha,\alpha'} \frac{1}{n_{xy}(\alpha,\alpha')} \right)^{-1} \end{aligned} \tag{3.6}$$

In order to solve (3.5) and (3.6), assuming they are solvable, we must be more explicit. Only the variables $\{n_x\}$ and $\{g_{xy}\}$ are really involved, and by virtue of (2.14) and a little algebra, (3.5) and (3.6) can be written as

$$Q_{xx}^2 + \sum_{\substack{\langle y,x\rangle\\ y>x}} Q_{xy}^2 = -\frac{q_x - 1}{1 - m_x^2} + \sum_{\langle y,x\rangle} \frac{1 - m_y^2}{1 - m_x^2 - m_y^2 - g_{xy}^2 + 2m_x m_y g_{xy}} \tag{3.7}$$

$$Q_{xy} Q_{yy} = \frac{g_{xy} - m_x m_y}{1 - m_x^2 - m_y^2 - g_{xy}^2 + 2m_x m_y g_{xy}} \tag{3.8}$$

But (3.7) can further be rewritten as

$$Q_{xx}^2 + \sum_{\substack{\langle y,x\rangle \\ y>x}} Q_{xy}^2 = \frac{1-m_{x'}^2}{1-m_x^2-m_{x'}^2-g_{xx'}^2+2m_x m_{x'} g_{xx'}} + \sum_{\substack{\langle y,x\rangle \\ y>x}} \frac{(g_{xy}-m_x m_y)^2}{(1-m_x^2)(1-m_x^2-m_y^2-g_{xy}^2+2m_x m_y g_{xy})} \tag{3.9}$$

Equations (3.8), (3.9) are clearly satisfied by

$$\begin{aligned} Q_{xx} &= \left[\frac{1-m_{x'}^2}{(1-m_x^2)(1-m_{x'}^2)-(g_{xx'}-m_x m_{x'})^2}\right]^{1/2} \\ Q_{yy} &= \left[\frac{1-m_x^2}{(1-m_x^2)(1-m_y^2)-(g_{xy}-m_x m_y)^2}\right]^{1/2} \\ Q_{xy} &= \frac{g_{xy}-m_x m_y}{(1-m_x^2)^{1/2}[(1-m_x^2)(1-m_y^2)-(g_{xy}-m_x m_y)^2]^{1/2}}, \qquad y>x \end{aligned} \tag{3.10}$$

and so our solution is complete. We can also return to the $n_A[\alpha_A]$ notation, in terms of which

$$\begin{gathered} Q_{yy} = \frac{1}{4}\left\{\frac{[\sum (1/n_{xy}(\alpha,\alpha'))]^2 - [\sum (\alpha/n_{xy}(\alpha,\alpha'))^2]}{\sum (1/n_{xy}(\alpha,\alpha'))}\right\}^{1/2} \\ (1-m_y^2) Q_{yy}^2 - (1-m_x^2) Q_{xy}^2 = 1 \end{gathered} \tag{3.11}$$

There is an immediate technical consequence of (3.3a). It is that the Jacobian of the $(h, J) \to (m, g)$ transformation can be evaluated very simply as

$$\operatorname{Det}\begin{pmatrix} \partial J/\partial g & \partial J/\partial m \\ \partial h/\partial g & \partial h/\partial m \end{pmatrix} = \prod_y Q_{yy}^2 \prod_{\langle x,y\rangle} \left(\frac{\partial J_{xy}}{\partial g_{xy}}\right) \tag{3.12}$$

readily transformed to

$$\begin{aligned} \operatorname{Jac}\left(\frac{J,h}{g,m}\right) &= \frac{\prod_x (n_x(+)\,n_x(-)/64)^{q_x-1}}{\prod_{\langle x,y\rangle} (n_{xy}(++)\,n_{xy}(+-)\,n_{xy}(-+)\,n_{xy}(--))} \\ &= \frac{\prod_{x,\alpha} (1+\alpha m_x)^{q_x-1}}{\prod_{\langle x,y\rangle} \prod_{\alpha,\alpha'} (1+\alpha m_x+\alpha' m_y+\alpha\alpha' g_x{}^y)} \end{aligned} \tag{3.13}$$

Application to spin-glass expectations is instantaneous. Suppose we have found the expression $A\{m_x, g_{xy}\}$ for an expectation or free energy in a fixed $\{h_x, J_{xy}\}$ thermal ensemble, and we now introduce the spin-glass weight factor $w\{h_x, J_{xy}\}$. Hence

$$\bar{A} = \int \cdots \int A\{m_x, g_{xy}\}\, w[h_x(m, g), J_{xy}(m, g)\}\prod dh_x \prod dJ_{xy}$$

$$= \int \cdots \int A\{m_x, g_{xy}\}\, w\{h_x[m, g], J_{xy}[m, g]\}$$

$$\times \operatorname{Jac}[J, h/g, m] \prod dm_x \prod dg_{xy}$$

and so, according to (3.13),

$$\bar{A} = \int \cdots \int A\{m_x, g_{xy}\}\, w\{h_y[m, g], J_{xy}[m, g]\}$$

$$\times \exp\left[\sum_{x,\alpha} (q_x - 1) \ln n_x(\alpha) - \sum_{\substack{\langle x, y\rangle \\ \alpha, \alpha'}} \ln n_{xy}(\alpha, \alpha')\right]$$

$$\times 2^{-6\Sigma(q_x - 1)} \prod dm_x \prod g_{xy} \tag{3.14}$$

But the computations involved in (3.14) need not be trivial.

4. DISCUSSION

We have succeeded in reexpressing the imposed $[h, J]$ distribution as a magnetization distribution

$$P\{m_x, g_{xy}\} = 2^{-6\Sigma(q_x - 1)} \exp\left[\sum_{x,\alpha} (q_x - 1) \ln n_x(\alpha) - \sum_{\substack{\langle x, y\rangle \\ \alpha, \alpha'}} \ln n_{xy}(\alpha, \alpha')\right]$$

$$\times w\{h_x[m, g], J_{xy}[m, g]\} \tag{4.1}$$

which is particularly appropriate, since the quantities to be averaged are almost always in neatest form as functions of the $[m, g]$. Furthermore, when the $[h, J]$ distribution is a product of independent components,

$$w\{h_x, J_{xy}\} = \prod_x f_x(h_x) \prod_{\langle x, y\rangle} f_{xy}(J_{xy}) \tag{4.2}$$

which is probably the best model of physical reality, the magnetization distribution, according to (2.22), decomposes as

$$P\{m_x, g_{xy}\} = \prod_{\langle x, y\rangle} Q_{xy}(m_x, g_{xy}, m_y) \prod_x R_x(m_x, \{g_{xy}\}) \tag{4.3}$$

where

$$Q_{xy}(m_x, g_{xy}, m_y) = \exp\left[-\sum_{\alpha,\alpha'} \ln n_{xy}(\alpha, \alpha') \right] f_{xy}\left(\tfrac{1}{4} \sum_{\alpha,\alpha'} \ln n_{xy}(\alpha, \alpha') \right)$$

$$R_x(m_x, \{g_{xy}\}) = \exp\left[(q_x - 1) \sum_{\alpha} \ln n_x(\alpha) \right] f_x \left(\tfrac{1}{4} \sum_{\substack{\langle y,x \rangle \\ \alpha,\alpha'}} \alpha \ln n_{xy}(\alpha, \alpha') - \tfrac{1}{2}(q_x - 1) \sum_{\alpha} \alpha \ln n_x(\alpha) \right) \tag{4.4}$$

This produces a recursion relation

$$L_{x \to y}(m_x, g_{xy}, m_y) = Q_{xy}(m_x, g_{xy}, m_y) \int R_y(m_y, \{g_{yz}\}) \prod_{z \neq x} L_{y \to z}(m_y, g_{yz}, m_z) \times \prod_{z \neq x} dm_z \prod_{z \neq y} dg_{yz} \tag{4.5}$$

for the "cumulative distribution" $L_{x \to y}$ on the y side of x, in terms of which simple expectations are simply expressed, e.g., the m_x probability distribution

$$\rho(m_x) = \int \prod_y L_{x \to y}(m_x, g_{xy}, m_y) \prod_y dm_y \prod_y dg_{xy} \tag{4.6}$$

In a homogeneous isotropic state, the recursion relation (4.5) collapses to a nonlinear integral equation[6]

$$L(m, g, m') = Q(m, g, m') \int R(m', g, \{g_z\}) \prod_{y=1}^{q-1} L(m', g_z, m_z) \prod_{1}^{q-1} dm_z\, dg_z \tag{4.7}$$

which can at least be solved numerically. Analytic solutions are available only in very special cases, but there are a few alternative strategies. The one we shall pursue here is that of expanding about the case of fixed couplings, and we will also restrict our attention to uniform systems. Thus in (4.2), we choose

$$\begin{aligned} f_x(h) &= (\lambda/\pi)^{1/2} \exp(-\lambda h^2) \\ f_{xy}(J) &= (\gamma/\pi)^{1/2} \exp[-\gamma(J - \bar{J})^2] \end{aligned} \tag{4.8}$$

so that $\lambda \to \infty$, $\gamma \to \infty$ would fix $h = 0$, $J = \bar{J}$, and according to (2.2) would correspond in the single-phase regime to

$$\bar{m} = 0, \qquad \bar{g} = \tanh \bar{J} \tag{4.9}$$

What we seek is the qualitative character of the resulting $[m, g]$ distribution, and for this purpose will make a Taylor expansion of (4.1), in the form

$$\ln P\{m_x, g_{xy}\} = \text{const} + (q-1)\sum_{x,\alpha} \ln n_x(\alpha) - \sum_{\substack{\langle x,y\rangle\\ \alpha,\alpha'}} \ln n_{xy}(\alpha,\alpha') - \lambda \sum_x h_x[m,g]^2 - \gamma \sum_{\langle x,y\rangle} (J_{xy}[m,g]-\bar{J})^2 \tag{4.10}$$

Setting $m_x = \delta m_x$, $g_{xy} = \bar{g} + \delta_{xy}$, we have

$$\begin{aligned} 2n_x(\alpha) &= 1 + \alpha\, \delta m_x \\ 4n_{xy}(\alpha,\alpha') &= (1+\alpha\alpha'\bar{g}) + \alpha\delta m_x + \alpha'\delta m_y + \alpha\alpha'\delta g_{xy} \end{aligned} \tag{4.11}$$

and so, to second order,

$$\begin{aligned} &(q-1)\sum_{x,\alpha} \ln n_x(\alpha) - \sum_{\substack{\langle x,y\rangle\\ \alpha,\alpha}} \ln n_{xy}(\alpha,\alpha') \\ &= \text{const} - \sum_x (q-1)\,\delta m_x^2 + \frac{4\bar{g}}{1-\bar{g}^2} \sum_{\{x,y\rangle} \delta g_{xy} + 2\frac{1+\bar{g}^2}{(1-\bar{g}^2)^2} \sum_{\langle x,y\rangle} (\delta g_{xy})^2 \\ &\quad + \sum_{\langle x,y\rangle} 2\frac{1+\bar{g}^2}{(1-\bar{g}^2)^2} (\delta m_x^2 + \delta m_y^2) - \sum_{\langle x,y\rangle} \frac{\gamma\bar{g}}{(1-\bar{g}^2)^2} \delta m_x\, \delta m_y \end{aligned} \tag{4.12}$$

On the other hand, we have to first order, from (2.22),

$$\begin{aligned} h_x &= -(q-1)\,\delta m_x - \frac{\bar{g}}{1-\bar{g}^2} \sum_{\langle y,x\rangle} \delta g_{xy} \\ J_{xy} - \bar{J} &= \frac{1}{1-\bar{g}^2} \sum_{\langle y,x\rangle} \delta g_{xy} \end{aligned} \tag{4.13}$$

so that, gathering terms,

$$\begin{aligned} &\delta \ln P\{m_x, g_{xy}\} \\ &= \frac{4\bar{g}}{1-\bar{g}^2} \sum_{\langle x,y\rangle} \delta g_{xy} + 2\frac{1+\bar{g}^2}{(1-\bar{g}^2)^2} \sum_{\langle x,y\rangle} (\delta g_{xy})^2 \\ &\quad - \frac{\gamma}{(1-\bar{g}_2)^2} \sum_x \left(\sum_{\langle y,x\rangle} \delta g_{xy} \right)^2 - \lambda \sum_x \left[(q-1)\,\delta m_x + \frac{\bar{g}}{1-\bar{g}^2} \sum_{\langle y,x\rangle} \delta g_{xy} \right]^2 \\ &\quad + \sum_{\langle x,y\rangle} \left\{ \left[2\frac{1+\bar{g}^2}{(1-\bar{g}^2)^2} - \frac{1}{2}\left(1-\frac{1}{q}\right) \right] (\delta m_x^2 + \delta m_y^2) - \frac{\gamma\bar{g}}{(1-\bar{g}^2)^2} \delta m_x\, \delta m_y \right\} \end{aligned} \tag{4.14}$$

Without going into further detail, we note the diagonal contribution of $\delta g_{xy} \equiv \delta g$:

$$[2(1+\bar{g}^2) - 2\gamma - 2\lambda \bar{g}^2] \left(\frac{\delta g}{1-\bar{g}^2} \right)^2 + 4\bar{g} \left(\frac{\delta g}{1-\bar{g}^2} \right) \tag{4.15}$$

which tells us that the distribution is stable only if $\lambda g^{-2} + \gamma > 1 + \bar{g}^2$, or

$$\lambda \sinh^2 \bar{J} + \gamma \cosh^2 \bar{J} > \cosh 2\bar{J} \tag{4.16}$$

Otherwise, and this could happen for the $q = 2$ one-dimensional lattice to the present order of approximation, a transition to another regime of correlations would take place. We know of course that a one-dimensional lattice with short-range interactions, spin glass or not, will have a unique distribution. Whether the above would result in any singularities in thermodynamic properties is not obvious, but we know of no proof that this cannot occur. Clearly, there is considerable qualitative information yet to be deduced.

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